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Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type polynomials

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available at the end of the article**Abstract**

In this paper, by considering Barnes-type Daehee polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

MSC: 05A15; 05A40; 11B68; 11B75; 65Q05**1 Introduction**

In this paper, we consider the polynomials $D_n^{(k)}(x|a_1, \dots, a_r)$ called the Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^x = \sum_{n=0}^{\infty} D_n^{(k)}(x|a_1, \dots, a_r) \frac{t^n}{n!}, \quad (1)$$

where $a_1, \dots, a_r \neq 0$. Here, $\text{Lif}_k(x)$ ($k \in \mathbb{Z}$) is the polyfactorial function [1] defined by

$$\text{Lif}_k(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!(m+1)^k}.$$

When $x = 0$, $D_n^{(k)}(a_1, \dots, a_r) = D_n^{(k)}(0|a_1, \dots, a_r)$ is called Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type number.

Recall that the Barnes-type Daehee polynomials of the first kind, denoted by $D_n(x|a_1, \dots, a_r)$, are given by the generating function

$$\prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^x = \sum_{n=0}^{\infty} D_n(x|a_1, \dots, a_r) \frac{t^n}{n!}.$$

If $a_1 = \dots = a_r = 1$, then $D_n^{(r)}(x) = D_n(x|\underbrace{1, \dots, 1}_r)$ are the Daehee polynomials of the first kind of order r . Daehee polynomials were defined by the second author [2] and have been investigated in [3, 4].

The poly-Cauchy polynomials of the first kind, denoted by $c_n^{(k)}(x)$ [5, 6], are given by the generating function as

$$\text{Lif}_k(\ln(1+t))(1+t)^x = \sum_{n=0}^{\infty} c_n^{(k)}(-x) \frac{t^n}{n!}.$$

In this paper, by considering Barnes-type Daehee polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let \mathbb{C} be the complex number field and let \mathcal{F} be the set of all formal power series in the variable t :

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbb{C} \right\}. \quad (2)$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . $\langle L|p(x) \rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on \mathbb{P}^* are defined by $\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle$, $\langle cL|p(x) \rangle = c\langle L|p(x) \rangle$, where c is a complex constant in \mathbb{C} . For $f(t) \in \mathcal{F}$, let us define the linear functional on \mathbb{P} by setting

$$\langle f(t)|x^n \rangle = a_n \quad (n \geq 0). \quad (3)$$

In particular,

$$\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \quad (4)$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$, we have $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} denotes both the algebra of formal power series in t and the vector space of all linear functionals on \mathbb{P} , and so an element $f(t)$ of \mathcal{F} will be thought of as both a formal power series and a linear functional. We call \mathcal{F} the *umbral algebra* and the *umbral calculus* is the study of the umbral algebra. The order $O(f(t))$ of a power series $f(t) (\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x) \rangle = n! \delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle \quad (5)$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!} \quad (6)$$

[7, Theorem 2.2.5]. Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x+y). \quad (7)$$

Sheffer sequences are characterized by the generating function [7, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim (g(t), f(t))$, we have the following equations [7, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \quad (8)$$

$$s_n(x) = \sum_{j=0}^n \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j, \quad (9)$$

$$s_n(x+y) = \sum_{j=0}^n \binom{n}{j} s_j(x) p_{n-j}(y), \quad (10)$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula [7, Corollary 3.8.2] is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, assume that

$$s_n(x) = \sum_{m=0}^n C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have [7, p.132]

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m | x^n \right\rangle. \quad (11)$$

3 Main results

From the definition (1), $D_n^{(k)}(x|a_1, \dots, a_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^r \left(\frac{e^{a_j t} - 1}{t} \right) \frac{1}{\text{Lif}_k(t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$D_n^{(k)}(x|a_1, \dots, a_r) \sim \left(\prod_{j=1}^r \left(\frac{e^{a_j t} - 1}{t} \right) \frac{1}{\text{Lif}_k(t)}, e^t - 1 \right). \quad (12)$$

3.1 Explicit expressions

Recall that Barnes' multiple Bernoulli polynomials $B_n(x|a_1, \dots, a_r)$ are defined by the generating function

$$\frac{t^r}{\prod_{j=1}^r (e^{a_j t} - 1)} e^{xt} = \sum_{n=0}^{\infty} B_n(x|a_1, \dots, a_r) \frac{t^n}{n!}, \quad (13)$$

where $a_1, \dots, a_r \neq 0$ [8, 9]. Let $(n)_j = n(n-1) \cdots (n-j+1)$ ($j \geq 1$) with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n, m)$ are defined by

$$(x)_n = \sum_{m=0}^n S_1(n, m) x^m.$$

Theorem 1

$$\begin{aligned} D_n^{(k)}(x|a_1, \dots, a_r) &= \sum_{m=0}^n \sum_{l=0}^m S_1(n, m) \frac{\binom{m}{l}}{(m-l+1)^k} B_l(x|a_1, \dots, a_r) \end{aligned} \quad (14)$$

$$= \sum_{j=0}^n \sum_{l=0}^{n-j} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} S_1(n-l, j) c_i^{(k)} D_{l-i}(a_1, \dots, a_r) x^j \quad (15)$$

$$= \sum_{l=0}^n \binom{n}{l} D_{n-l}(a_1, \dots, a_r) c_l^{(k)}(-x) \quad (16)$$

$$= \sum_{l=0}^n \binom{n}{l} c_{n-l}^{(k)} D_l(x|a_1, \dots, a_r). \quad (17)$$

Proof Since

$$\prod_{j=1}^r \left(\frac{e^{a_j t} - 1}{t} \right) \frac{1}{\text{Lif}_k(t)} D_n^{(k)}(x|a_1, \dots, a_r) \sim (1, e^t - 1) \quad (18)$$

and

$$(x)_n \sim (1, e^t - 1), \quad (19)$$

we have

$$\begin{aligned}
 D_n^{(k)}(x|a_1, \dots, a_r) &= \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) \text{Lif}_k(t)(x)_n \\
 &= \sum_{m=0}^n S_1(n, m) \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) \text{Lif}_k(t)x^m \\
 &= \sum_{m=0}^n S_1(n, m) \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) \sum_{l=0}^m \frac{t^l}{l!(l+1)^k} x^{m-l} \\
 &= \sum_{m=0}^n S_1(n, m) \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) \sum_{l=0}^m \frac{(m)_l}{l!(l+1)^k} x^{m-l} \\
 &= \sum_{m=0}^n S_1(n, m) \sum_{l=0}^m \frac{(m)_l}{l!(l+1)^k} \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) x^{m-l} \\
 &= \sum_{m=0}^n \sum_{l=0}^m S_1(n, m) \frac{(m)_l}{(l+1)^k} B_{m-l}(x|a_1, \dots, a_r) \\
 &= \sum_{m=0}^n \sum_{l=0}^m S_1(n, m) \frac{(m)_l}{(m-l+1)^k} B_l(x|a_1, \dots, a_r).
 \end{aligned}$$

Thus, we get (14).

By (9) with (12), we get

$$\begin{aligned}
 &\langle g(\tilde{f}(t))^{-1} \tilde{f}(t)^j | x^n \rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^j \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| \sum_{l=0}^{\infty} \frac{j!}{(l+j)!} S_1(l+j, j) t^{l+j} x^n \right\rangle \\
 &= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j, j) (n)_{l+j} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| x^{n-l-j} \right\rangle \\
 &= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j, j) \left\langle \sum_{i=0}^{\infty} D_i^{(k)}(a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^{n-l-j} \right\rangle \\
 &= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j, j) D_{n-l-j}^{(k)}(a_1, \dots, a_r) \\
 &= \sum_{l=0}^{n-j} j! \binom{n}{l} S_1(n-l, j) D_l^{(k)}(a_1, \dots, a_r).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 &\langle g(\tilde{f}(t))^{-1} \tilde{f}(t)^j | x^n \rangle \\
 &= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j, j) (n)_{l+j} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) x^{n-l-j} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j, j) \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left| \sum_{i=0}^{n-l-j} c_i^{(k)} \frac{t^i}{i!} x^{n-l-j} \right. \right\rangle \\
&= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j, j) \sum_{i=0}^{n-l-j} c_i^{(k)} \frac{(n-l-j)_i}{i!} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left| x^{n-l-j-i} \right. \right\rangle \\
&= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j, j) \sum_{i=0}^{n-l-j} c_i^{(k)} \frac{(n-l-j)_i}{i!} \left\langle \sum_{m=0}^{\infty} D_m(a_1, \dots, a_r) \frac{t^m}{m!} \left| x^{n-l-j-i} \right. \right\rangle \\
&= \sum_{l=0}^{n-j} \sum_{i=0}^{n-l-j} j! \binom{n}{l+j} \binom{n-l-j}{i} S_1(l+j, j) c_i^{(k)} D_{n-l-j-i}(a_1, \dots, a_r) \\
&= \sum_{l=0}^{n-j} \sum_{i=0}^l j! \binom{n}{l} \binom{l}{i} S_1(n-l, j) c_i^{(k)} D_{l-i}(a_1, \dots, a_r).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&D_n^{(k)}(x|a_1, \dots, a_r) \\
&= \sum_{j=0}^n \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l, j) D_l^{(k)}(a_1, \dots, a_r) x^j \\
&= \sum_{j=0}^n \sum_{l=0}^{n-j} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} S_1(n-l, j) c_i^{(k)} D_{l-i}(a_1, \dots, a_r) x^j,
\end{aligned}$$

which is the identity (15).

Next,

$$\begin{aligned}
D_n^{(k)}(y|a_1, \dots, a_r) &= \left\langle \sum_{i=0}^{\infty} D_i^{(k)}(y|a_1, \dots, a_r) \frac{t^i}{i!} \left| x^n \right. \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^y \left| x^n \right. \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left| \text{Lif}_k(\ln(1+t))(1+t)^y x^n \right. \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left| \sum_{l=0}^n c_l^{(k)} (-y) \frac{t^l}{l!} x^n \right. \right\rangle \\
&= \sum_{l=0}^n \binom{n}{l} c_l^{(k)} (-y) \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left| x^{n-l} \right. \right\rangle \\
&= \sum_{l=0}^n \binom{n}{l} c_l^{(k)} (-y) \left\langle \sum_{i=0}^{\infty} D_i(a_1, \dots, a_r) \frac{t^i}{i!} \left| x^{n-l} \right. \right\rangle \\
&= \sum_{l=0}^n \binom{n}{l} c_l^{(k)} (-y) D_{n-l}(a_1, \dots, a_r).
\end{aligned}$$

Thus, we obtain (16).

Finally, we obtain

$$\begin{aligned}
 D_n^{(k)}(y|a_1, \dots, a_r) &= \left\langle \sum_{i=0}^{\infty} D_i^{(k)}(y|a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^y \middle| x^n \right\rangle \\
 &= \left\langle \text{Lif}_k(\ln(1+t)) \middle| \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y x^n \right\rangle \\
 &= \left\langle \text{Lif}_k(\ln(1+t)) \middle| \sum_{l=0}^n D_l(y|a_1, \dots, a_r) \frac{t^l}{l!} x^n \right\rangle \\
 &= \sum_{l=0}^n D_l(y|a_1, \dots, a_r) \binom{n}{l} (\text{Lif}_k(\ln(1+t)) | x^{n-l}) \\
 &= \sum_{l=0}^n D_l(y|a_1, \dots, a_r) \binom{n}{l} \left\langle \sum_{i=0}^{\infty} c_i^{(k)} \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} D_l(y|a_1, \dots, a_r) c_{n-l}^{(k)}.
 \end{aligned}$$

Thus, we get the identity (17). \square

3.2 Sheffer identity

Theorem 2

$$D_n^{(k)}(x + y|a_1, \dots, a_r) = \sum_{j=0}^n \binom{n}{j} D_j^{(k)}(x|a_1, \dots, a_r) (y)_{n-j}. \quad (20)$$

Proof By (12) with

$$\begin{aligned}
 p_n(x) &= \prod_{j=1}^r \left(\frac{e^{a_j t} - 1}{t} \right) \frac{1}{\text{Lif}_k(t)} D_n(x|a_1, \dots, a_r) \\
 &= (x)_n \sim (1, e^t - 1),
 \end{aligned}$$

using (10), we have (20). \square

3.3 Difference relations

Theorem 3

$$D_n^{(k)}(x + 1|a_1, \dots, a_r) - D_n^{(k)}(x|a_1, \dots, a_r) = n D_{n-1}^{(k)}(x|a_1, \dots, a_r). \quad (21)$$

Proof By (8) with (12), we get

$$(e^t - 1) D_n^{(k)}(x|a_1, \dots, a_r) = n D_{n-1}^{(k)}(x|a_1, \dots, a_r).$$

By (7), we have (21). \square

3.4 Recurrence

Theorem 4

$$\begin{aligned}
 D_{n+1}^{(k)}(x|a_1, \dots, a_r) &= xD_n^{(k)}(x-1|a_1, \dots, a_r) \\
 &\quad - \sum_{m=0}^n \sum_{j=1}^r \sum_{l=1}^{m+1} \sum_{i=0}^l \frac{\binom{m+1}{l} \binom{l}{i}}{(m+1)(l-i+1)^k} S_1(n, m) \\
 &\quad \times (-a_j)^{m+1-l} B_{m+1-l} B_i(x-1|a_1, \dots, a_r) \\
 &\quad + \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n, m) B_l(x-1|a_1, \dots, a_r),
 \end{aligned} \tag{22}$$

where B_n is the n th ordinary Bernoulli number.

Proof By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x) \tag{23}$$

[7, Corollary 3.7.2] with (12), we get

$$D_{n+1}^{(k)}(x|a_1, \dots, a_r) = xD_n^{(k)}(x-1|a_1, \dots, a_r) - e^{-t} \frac{g'(t)}{g(t)} D_n^{(k)}(x|a_1, \dots, a_r).$$

Now,

$$\begin{aligned}
 \frac{g'(t)}{g(t)} &= (\ln g(t))' \\
 &= \left(\sum_{j=1}^r \ln(e^{a_j t} - 1) - r \ln t - \ln \text{Lif}_k(t) \right)' \\
 &= \sum_{j=1}^r \frac{a_j e^{a_j t}}{e^{a_j t} - 1} - \frac{r}{t} - \frac{\text{Lif}'_k(t)}{\text{Lif}_k(t)} \\
 &= \frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1) (a_j t e^{a_j t} - e^{a_j t} + 1)}{t \prod_{j=1}^r (e^{a_j t} - 1)} - \frac{\text{Lif}'_k(t)}{\text{Lif}_k(t)}.
 \end{aligned}$$

Observe that

$$\begin{aligned}
 &\frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1) (a_j t e^{a_j t} - e^{a_j t} + 1)}{\prod_{j=1}^r (e^{a_j t} - 1)} \\
 &= \frac{\frac{1}{2} (\sum_{j=1}^r a_1 \cdots a_{j-1} a_j^2 a_{j+1} \cdots a_r) t^{r+1} + \cdots}{(a_1 \cdots a_r) t^r + \cdots} \\
 &= \frac{1}{2} \left(\sum_{j=1}^r a_j \right) t + \cdots
 \end{aligned}$$

is a series with order ≥ 1 . Since

$$D_n^{(k)}(x|a_1, \dots, a_r) = \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) \text{Lif}_k(t)(x)_n = \sum_{m=0}^n S_1(n, m) \prod_{j=1}^r \left(\frac{t}{e^{a_j t} - 1} \right) \text{Lif}_k(t)x^m,$$

we have

$$\begin{aligned} \frac{g'(t)}{g(t)} D_n^{(k)}(x|a_1, \dots, a_r) &= \sum_{m=0}^n S_1(n, m) \frac{g'(t)}{g(t)} \left(\prod_{j=1}^r \frac{t}{e^{a_j t} - 1} \right) \text{Lif}_k(t)x^m \\ &= \sum_{m=0}^n S_1(n, m) \text{Lif}_k(t) \left(\prod_{j=1}^r \frac{t}{e^{a_j t} - 1} \right) \\ &\quad \times \frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{t \prod_{j=1}^r (e^{a_j t} - 1)} x^m \\ &\quad - \sum_{m=0}^n S_1(n, m) \text{Lif}'_k(t) \left(\prod_{j=1}^r \frac{t}{e^{a_j t} - 1} \right) x^m. \end{aligned} \quad (24)$$

Since

$$\begin{aligned} &\frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{t \prod_{j=1}^r (e^{a_j t} - 1)} x^m \\ &= \frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{\prod_{j=1}^r (e^{a_j t} - 1)} \frac{x^{m+1}}{m+1} \\ &= \frac{1}{m+1} \sum_{j=1}^r \left(\frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - 1 \right) x^{m+1} \\ &= \frac{1}{m+1} \sum_{j=1}^r \left(\sum_{l=0}^{\infty} \frac{(-1)^l B_l a_j^l}{l!} t^l - 1 \right) x^{m+1} \\ &= \frac{1}{m+1} \sum_{j=1}^r \left(\sum_{l=0}^{m+1} \binom{m+1}{l} (-a_j)^l B_l x^{m+1-l} - x^{m+1} \right) \\ &= \frac{1}{m+1} \sum_{j=1}^r \sum_{l=1}^{m+1} \binom{m+1}{l} (-a_j)^l B_l x^{m+1-l} \\ &= \frac{1}{m+1} \sum_{j=1}^r \sum_{l=1}^{m+1} \binom{m+1}{l} (-a_j)^{m+1-l} B_{m+1-l} x^l, \end{aligned}$$

the first term in (24) is

$$\begin{aligned} &\sum_{m=0}^n \frac{S_1(n, m)}{m+1} \sum_{j=1}^r \sum_{l=1}^{m+1} \binom{m+1}{l} (-a_j)^{m+1-l} B_{m+1-l} \text{Lif}_k(t) \left(\prod_{j=1}^r \frac{t}{e^{a_j t} - 1} \right) x^l \\ &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \sum_{j=1}^r \sum_{l=1}^{m+1} \binom{m+1}{l} (-a_j)^{m+1-l} B_{m+1-l} \sum_{i=0}^l \frac{t^i}{i!(i+1)^k} B_l(x|a_1, \dots, a_r) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \sum_{j=1}^r \sum_{l=1}^{m+1} \binom{m+1}{l} (-a_j)^{m+1-l} B_{m+1-l} \sum_{i=0}^l \frac{\binom{l}{i}}{(i+1)^k} B_{l-i}(x|a_1, \dots, a_r) \\
 &= \sum_{m=0}^n \sum_{j=1}^r \sum_{l=1}^{m+1} \sum_{i=0}^l \frac{\binom{m+1}{l} \binom{l}{i}}{(m+1)(l-i+1)^k} S_1(n, m) (-a_j)^{m+1-l} B_{m+1-l} B_i(x|a_1, \dots, a_r).
 \end{aligned}$$

Since

$$\text{Lif}_{k-1}(t) - \text{Lif}_k(t) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^k} \right) t + \dots, \quad (25)$$

the second term in (24) is

$$\begin{aligned}
 &\sum_{m=0}^n S_1(n, m) \frac{\text{Lif}_{k-1}(t) - \text{Lif}_k(t)}{t} B_m(x|a_1, \dots, a_r) \\
 &= \sum_{m=0}^n S_1(n, m) (\text{Lif}_{k-1}(t) - \text{Lif}_k(t)) \frac{B_{m+1}(x|a_1, \dots, a_r)}{m+1} \\
 &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} (\text{Lif}_{k-1}(t) - \text{Lif}_k(t)) B_{m+1}(x|a_1, \dots, a_r) \\
 &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^{k-1}} B_{m+1}(x|a_1, \dots, a_r) - \sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^k} B_{m+1}(x|a_1, \dots, a_r) \right) \\
 &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^{k-1}} B_{m+1-l}(x|a_1, \dots, a_r) - \sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^k} B_{m+1-l}(x|a_1, \dots, a_r) \right) \\
 &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \sum_{l=1}^{m+1} \frac{\binom{m+1}{l} l}{(l+1)^k} B_{m+1-l}(x|a_1, \dots, a_r) \\
 &= \sum_{m=0}^n \sum_{l=1}^{m+1} \binom{m}{l-1} S_1(n, m) \frac{1}{(l+1)^k} B_{m+1-l}(x|a_1, \dots, a_r) \\
 &= \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n, m) B_l(x|a_1, \dots, a_r).
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 D_{n+1}^{(k)}(x|a_1, \dots, a_r) &= x D_n^{(k)}(x-1|a_1, \dots, a_r) \\
 &\quad - \sum_{m=0}^n \sum_{j=1}^r \sum_{l=1}^{m+1} \sum_{i=0}^l \frac{\binom{m+1}{l} \binom{l}{i}}{(m+1)(l-i+1)^k} S_1(n, m) \\
 &\quad \times (-a_j)^{m+1-l} B_{m+1-l} B_i(x-1|a_1, \dots, a_r) \\
 &\quad + \sum_{m=0}^n \sum_{l=0}^m \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n, m) B_l(x-1|a_1, \dots, a_r),
 \end{aligned}$$

which is the identity (22). \square

3.5 Differentiation

Theorem 5

$$\frac{d}{dx} D_n^{(k)}(x|a_1, \dots, a_r) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} D_l^{(k)}(x|a_1, \dots, a_r). \quad (26)$$

Proof We shall use

$$\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t) | x^{n-l} \rangle s_l(x)$$

(cf. [7, Theorem 2.3.12]). Since

$$\begin{aligned} \langle \bar{f}(t) | x^{n-l} \rangle &= \langle \ln(1+t) | x^{n-l} \rangle \\ &= \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1} t^m}{m} \middle| x^{n-l} \right\rangle \\ &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} \langle t^m | x^{n-l} \rangle \\ &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} (n-l)! \delta_{m,n-l} \\ &= (-1)^{n-l-1} (n-l-1)!, \end{aligned}$$

with (12), we have

$$\begin{aligned} \frac{d}{dx} D_n^{(k)}(x|a_1, \dots, a_r) &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1} (n-l-1)! D_l^{(k)}(x|a_1, \dots, a_r) \\ &= n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} D_l^{(k)}(x|a_1, \dots, a_r), \end{aligned}$$

which is the identity (26). □

3.6 One more relation

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

(see e.g. [1, 10]).

Theorem 6

$$\begin{aligned} D_n^{(k)}(x|a_1, \dots, a_r) &= x D_{n-1}^{(k)}(x-1|a_1, \dots, a_r) \\ &\quad + \frac{1}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k-1)}(x-1|a_1, \dots, a_r) \end{aligned}$$

$$\begin{aligned}
 & + \frac{r-1}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1, \dots, a_r) \\
 & - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1, \dots, a_r, a_j). \tag{27}
 \end{aligned}$$

Proof For $n \geq 1$, we have

$$\begin{aligned}
 D_n^{(k)}(y|a_1, \dots, a_r) &= \left\langle \sum_{l=0}^{\infty} D_l^{(k)}(y|a_1, \dots, a_r) \frac{t^l}{l!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t))(1+t)^y \middle| x^n \right\rangle \\
 &= \left\langle \partial_t \left(\prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t))(1+t)^y \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \left(\partial_t \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \right) \text{Lif}_k(\ln(1+t))(1+t)^y \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (\partial_t \text{Lif}_k(\ln(1+t)))(1+t)^y \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (\partial_t (1+t)^y) \middle| x^{n-1} \right\rangle.
 \end{aligned}$$

The third term is

$$\begin{aligned}
 & y \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
 &= y D_{n-1}^{(k)}(y-1|a_1, \dots, a_r).
 \end{aligned}$$

By (25), the second term is

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \frac{\text{Lif}_{k-1}(\ln(1+t)) - \text{Lif}_k(\ln(1+t))}{(1+t) \ln(1+t)} (1+t)^y \middle| x^{n-1} \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \frac{\text{Lif}_{k-1}(\ln(1+t)) - \text{Lif}_k(\ln(1+t))}{t} (1+t)^{y-1} \middle| \frac{t}{\ln(1+t)} x^{n-1} \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \frac{\text{Lif}_{k-1}(\ln(1+t)) - \text{Lif}_k(\ln(1+t))}{t} (1+t)^{y-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^{n-1} \right\rangle \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} c_l \\
 &\quad \times \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{y-1} \middle| \frac{\text{Lif}_{k-1}(\ln(1+t)) - \text{Lif}_k(\ln(1+t))}{t} x^{n-1-l} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{n-1} \binom{n-1}{l} c_l \\
&\quad \times \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{y-1} \left| \left(\text{Lif}_{k-1}(\ln(1+t)) - \text{Lif}_k(\ln(1+t)) \right) \frac{x^{n-l}}{n-l} \right| \right\rangle \\
&= \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l \left(\left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_{k-1}(\ln(1+t)) (1+t)^{y-1} \right| x^{n-l} \right\rangle \right. \\
&\quad \left. - \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y-1} \right| x^{n-l} \right\rangle \right) \\
&= \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l (D_{n-l}^{(k-1)}(y-1|a_1, \dots, a_r) - D_{n-l}^{(k)}(y-1|a_1, \dots, a_r)).
\end{aligned}$$

Since

$$\partial_t \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) = \frac{1}{1+t} \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \frac{\sum_{i=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1} \right)}{t},$$

with

$$\sum_{i=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1} \right) = -\frac{1}{2} \left(\sum_{i=1}^r a_i \right) t + \dots$$

a series with order (≥ 1), the first term is

$$\begin{aligned}
&\left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y-1} \right| \frac{\sum_{i=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1} \right)}{t} x^{n-1} \right\rangle \\
&= \frac{1}{n} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y-1} \right| \sum_{i=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1} \right) x^n \right\rangle \\
&= \frac{r}{n} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y-1} \right| \frac{t}{\ln(1+t)} x^n \right\rangle \\
&\quad - \frac{1}{n} \sum_{i=1}^r a_i \left\langle \frac{\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y+a_i-1} \right| \frac{t}{\ln(1+t)} x^n \right\rangle \\
&= \frac{r}{n} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y-1} \right| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
&\quad - \frac{1}{n} \sum_{i=1}^r a_i \left\langle \frac{\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y+a_i-1} \right| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
&= \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{y-1} \right| x^{n-l} \right\rangle \\
&\quad - \frac{1}{n} \sum_{i=1}^r a_i \sum_{l=0}^n \binom{n}{l} c_l
\end{aligned}$$

$$\begin{aligned} & \times \left\langle \frac{\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{y+a_i-1} \middle| x^{n-l} \right\rangle \\ &= \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k)}(y-1|a_1, \dots, a_r) \\ & \quad - \frac{1}{n} \sum_{i=1}^r a_i \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k)}(y+a_i-1|a_1, \dots, a_r, a_i). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} D_n^{(k)}(x|a_1, \dots, a_r) &= x D_{n-1}^{(k)}(x-1|a_1, \dots, a_r) \\ & \quad + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l (D_{n-l}^{(k-1)}(x-1|a_1, \dots, a_r) - D_{n-l}^{(k)}(x-1|a_1, \dots, a_r)) \\ & \quad + \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1, \dots, a_r) \\ & \quad - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1, \dots, a_r, a_j) \\ &= x D_{n-1}^{(k)}(x-1|a_1, \dots, a_r) \\ & \quad + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l D_{n-l}^{(k-1)}(x-1|a_1, \dots, a_r) \\ & \quad + \frac{r-1}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1, \dots, a_r) \\ & \quad + \frac{1}{n} c_n - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1, \dots, a_r, a_j) \\ &= x D_{n-1}^{(k)}(x-1|a_1, \dots, a_r) + \frac{1}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k-1)}(x-1|a_1, \dots, a_r) \\ & \quad + \frac{r-1}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1, \dots, a_r) \\ & \quad - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1, \dots, a_r, a_j), \end{aligned}$$

which is the identity (27). □

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 For $n \geq m \geq 1$, we have

$$\begin{aligned} & m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) D_l^{(k)}(a_1, \dots, a_r) \\ &= \frac{mr}{n} \sum_{l=0}^{n-m} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} S_1(n-l, m) c_{l-i} D_i^{(k)}(-1|a_1, \dots, a_r) \end{aligned}$$

$$\begin{aligned}
& -\frac{m}{n} \sum_{l=0}^{n-m} \sum_{i=0}^l \sum_{j=1}^r \binom{n}{l} \binom{l}{i} S_1(n-l, m) a_j c_{l-i} D_i^{(k)}(a_j - 1 | a_1, \dots, a_r, a_j) \\
& + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k-1)}(-1 | a_1, \dots, a_r) \\
& + (m-1) \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k)}(-1 | a_1, \dots, a_r). \tag{28}
\end{aligned}$$

Proof We shall compute

$$\left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| x^n \right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
& \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| x^n \right\rangle \\
& = \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| (\ln(1+t))^m x^n \right\rangle \\
& = \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| \sum_{l=0}^{\infty} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^n \right\rangle \\
& = \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m, m) (n)_{l+m} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| x^{n-l-m} \right\rangle \\
& = \sum_{l=0}^{n-m} m! \binom{n}{l+m} S_1(l+m, m) D_{n-l-m}^{(k)}(a_1, \dots, a_r) \\
& = \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l, m) D_l^{(k)}(a_1, \dots, a_r).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| x^n \right\rangle \\
& = \left\langle \partial_t \left(\prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \right) \middle| x^{n-1} \right\rangle \\
& = \left\langle \left(\partial_t \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| x^{n-1} \right\rangle \\
& \quad + \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\partial_t \text{Lif}_k(\ln(1+t))) (\ln(1+t))^m \middle| x^{n-1} \right\rangle \\
& \quad + \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\partial_t (\ln(1+t))^m) \middle| x^{n-1} \right\rangle. \tag{29}
\end{aligned}$$

The third term of (29) is equal to

$$\begin{aligned}
 & m \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{-1} \left| (\ln(1+t))^{m-1} x^{n-1} \right. \right\rangle \\
 &= m \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{-1} \left| \right. \right. \\
 &\quad \left. \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) t^{l+m-1} x^{n-1} \right\rangle \\
 &= m \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) (n-1)_{l+m-1} \\
 &\quad \times \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{-1} \left| x^{n-l-m} \right. \right\rangle \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l+m-1} S_1(l+m-1, m-1) D_{n-l-m}^{(k)}(-1|a_1, \dots, a_r) \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k)}(-1|a_1, \dots, a_r).
 \end{aligned}$$

The second term of (29) is equal to

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left(\frac{\text{Lif}_{k-1}(\ln(1+t)) - \text{Lif}_k(\ln(1+t))}{(1+t) \ln(1+t)} \right) (\ln(1+t))^m \left| x^{n-1} \right. \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_{k-1}(\ln(1+t))(1+t)^{-1} \left| (\ln(1+t))^{m-1} x^{n-1} \right. \right\rangle \\
 &\quad - \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{-1} \left| (\ln(1+t))^{m-1} x^{n-1} \right. \right\rangle \\
 &= (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k-1)}(-1|a_1, \dots, a_r) \\
 &\quad - (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k)}(-1|a_1, \dots, a_r).
 \end{aligned}$$

The first term of (29) is equal to

$$\begin{aligned}
 & \left\langle \frac{1}{1+t} \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \frac{\sum_{j=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \left| x^{n-1} \right. \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{-1} (\ln(1+t))^m \left| \right. \right. \\
 &\quad \left. \frac{\sum_{j=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^{n-1} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{-1} (\ln(1+t))^m \right| \\
&\quad \sum_{j=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) x^n \Bigg\rangle \\
&= \frac{1}{n} \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{-1} \right. \\
&\quad \times \sum_{j=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \Big| (\ln(1+t))^m x^n \Bigg\rangle \\
&= \frac{1}{n} \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{-1} \right. \\
&\quad \times \sum_{j=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \Big| \sum_{l=0}^{\infty} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^n \Bigg\rangle \\
&= \frac{1}{n} \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m, m) (n)_{l+m} \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{-1} \right. \\
&\quad \times \sum_{j=1}^r \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \Big| x^{n-l-m} \Bigg\rangle \\
&= \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m, m) \\
&\quad \times \left(r \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{-1} \right| \frac{t}{\ln(1+t)} x^{n-l-m} \right\rangle \right. \\
&\quad \left. - \sum_{j=1}^r a_j \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{a_j-1} \right| \right. \\
&\quad \left. \frac{t}{\ln(1+t)} x^{n-l-m} \right\rangle \Bigg) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m, m) \\
&\quad \times \left(r \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{-1} \right| \sum_{v=0}^{\infty} c_v \frac{t^v}{v!} x^{n-l-m} \right\rangle \right. \\
&\quad \left. - \sum_{j=1}^r a_j \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \text{Lif}_k(\ln(1+t)) (1+t)^{a_j-1} \right| \right. \\
&\quad \left. \sum_{v=0}^{\infty} c_v \frac{t^v}{v!} x^{n-l-m} \right\rangle \Bigg) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m, m)
\end{aligned}$$

$$\begin{aligned}
& \times \left(r \sum_{v=0}^{n-l-m} \binom{n-l-m}{v} c_v \left\langle \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{-1} \middle| x^{n-l-m-v} \right\rangle \right. \\
& - \sum_{j=1}^r a_j \sum_{v=0}^{n-l-m} \binom{n-l-m}{v} c_v \\
& \times \left. \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) \text{Lif}_k(\ln(1+t))(1+t)^{a_j-1} \middle| x^{n-l-m-v} \right\rangle \right) \\
& = \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m, m) \\
& \times \left(r \sum_{v=0}^{n-l-m} \binom{n-l-m}{v} c_v D_{n-l-m-v}^{(k)}(-1|a_1, \dots, a_r) \right. \\
& - \sum_{j=1}^r \sum_{v=0}^{n-l-m} \binom{n-l-m}{v} a_j c_v D_{n-l-m-v}^{(k)}(a_j-1|a_1, \dots, a_r, a_j) \Big) \\
& = \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \\
& \times \left(r \sum_{i=0}^l \binom{l}{i} c_i D_{n-i}^{(k)}(-1|a_1, \dots, a_r) - \sum_{j=1}^r \sum_{i=0}^l \binom{l}{i} a_j c_i D_{l-i}^{(k)}(a_j-1|a_1, \dots, a_r, a_j) \right).
\end{aligned}$$

Therefore, we get for $n \geq m \geq 1$

$$\begin{aligned}
& m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) D_l^{(k)}(a_1, \dots, a_r) \\
& = m! \frac{r}{n} \sum_{l=0}^{n-m} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} S_1(n-l, m) c_i D_{l-i}^{(k)}(-1|a_1, \dots, a_r) \\
& - m! \frac{1}{n} \sum_{l=0}^{n-m} \sum_{i=0}^l \sum_{j=1}^r \binom{n}{l} \binom{l}{i} S_1(n-l, m) a_j c_i D_{l-i}^{(k)}(a_j-1|a_1, \dots, a_r, a_j) \\
& + (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k-1)}(-1|a_1, \dots, a_r) \\
& - (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k)}(-1|a_1, \dots, a_r) \\
& + m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k)}(-1|a_1, \dots, a_r).
\end{aligned}$$

Dividing both sides by $(m-1)!$, we obtain for $n \geq m \geq 1$

$$\begin{aligned}
& m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) D_l^{(k)}(a_1, \dots, a_r) \\
& = \frac{mr}{n} \sum_{l=0}^{n-m} \sum_{i=0}^l \binom{n}{l} \binom{l}{i} S_1(n-l, m) c_i D_{l-i}^{(k)}(-1|a_1, \dots, a_r)
\end{aligned}$$

$$\begin{aligned}
 & - \frac{m}{n} \sum_{l=0}^{n-m} \sum_{i=0}^l \sum_{j=1}^r \binom{n}{l} \binom{l}{i} S_1(n-l, m) a_j c_{l-i} D_i^{(k)}(a_j - 1 | a_1, \dots, a_r, a_j) \\
 & + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k-1)}(-1 | a_1, \dots, a_r) \\
 & + (m-1) \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l^{(k)}(-1 | a_1, \dots, a_r).
 \end{aligned}$$

Thus, we get (28). \square

3.8 A relation with the falling factorials

Theorem 8

$$D_n^{(k)}(x | a_1, \dots, a_r) = \sum_{m=0}^n \binom{n}{m} D_{n-m}^{(k)}(a_1, \dots, a_r) (x)_m. \quad (30)$$

Proof For (12) and (19), assume that $D_n^{(k)}(x | a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m}(x)_m$. By (11), we have

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^r \left(\frac{e^{a_j \ln(1+t)} - 1}{\ln(1+t)} \right) \frac{1}{\text{Lif}_k(\ln(1+t))}} t^m \middle| x^n \right\rangle \\
 &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| t^m x^n \right\rangle \\
 &= \binom{n}{m} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) \middle| x^{n-m} \right\rangle \\
 &= \binom{n}{m} D_{n-m}^{(k)}(a_1, \dots, a_r).
 \end{aligned}$$

Thus, we get the identity (30). \square

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r , $H_n^{(r)}(x | \lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t - \lambda} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x | \lambda) \frac{t^n}{n!}$$

(see e.g. [11]).

Theorem 9

$$\begin{aligned}
 D_n^{(k)}(x | a_1, \dots, a_r) &= \sum_{m=0}^n \left(\sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j \right. \\
 &\quad \left. \times (1-\lambda)^{-j} S_1(n-j-l, m) D_l^{(k)}(a_1, \dots, a_r) \right) H_m^{(s)}(x | \lambda). \quad (31)
 \end{aligned}$$

Proof For (12) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda} \right)^s, t \right), \quad (32)$$

assume that $D_n^{(k)}(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\lambda)$. By (11), similarly to the proof of (28), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)} - \lambda}{1 - \lambda} \right)^s}{\prod_{j=1}^r \left(\frac{e^{a_j \ln(1+t)} - 1}{\ln(1+t)} \right) \frac{1}{\text{Lif}_k(\ln(1+t))}} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m (1-\lambda+t)^s \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \\ &\quad \times \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| \sum_{i=0}^{\min\{s,n\}} \binom{s}{i} (1-\lambda)^{s-i} t^i x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \\ &\quad \times \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| x^{n-i} \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) D_l^{(k)}(a_1, \dots, a_r) \\ &= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{s}{i} \binom{n-i}{l} (n)_i (1-\lambda)^{-i} S_1(n-i-l, m) D_l^{(k)}(a_1, \dots, a_r). \end{aligned}$$

Thus, we get the identity (31). \square

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(r)}(x)}{n!} t^n$$

(see e.g. [7, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)} \right)^r = \sum_{n=0}^{\infty} \frac{\mathfrak{C}_n^{(r)}}{n!} t^n$$

(see e.g. [12, (2.1)], [13, (6)]).

Theorem 10

$$D_n^{(k)}(x|a_1, \dots, a_r) = \sum_{m=0}^n \left(\sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) D_l^{(k)}(a_1, \dots, a_r) \right) \mathfrak{B}_m^{(s)}(x). \quad (33)$$

Proof For (12) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right), \quad (34)$$

assume that $D_n^{(k)}(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m} \mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (28), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)} - 1}{\ln(1+t)} \right)^s}{\prod_{j=1}^r \left(\frac{e^{a_j \ln(1+t)} - 1}{\ln(1+t)} \right) \frac{1}{\text{Lif}_k(\ln(1+t))}} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| \left(\frac{t}{\ln(1+t)} \right)^s x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)} \frac{t^i}{i!} x^n \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \text{Lif}_k(\ln(1+t)) (\ln(1+t))^m \middle| x^{n-i} \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) D_l^{(k)}(a_1, \dots, a_r) \\ &= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) D_l^{(k)}(a_1, \dots, a_r). \end{aligned}$$

Thus, we get the identity (33). \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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